

Global weak solutions for coupled transport processes in concrete walls at high temperatures

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Abstract

We consider an initial-boundary value problem for a fully nonlinear coupled parabolic system with nonlinear boundary conditions modelling hygro-thermal behavior of concrete at high temperatures. We prove a global existence of a weak solution to this system on an arbitrary time interval. The main result is proved by an approximation procedure. This consists in proving the existence of solutions to mollified problems using the Leray-Schauder theorem, for which a priori estimates are obtained. The limit then provides a weak solution for the original problem. A practical example illustrates a performance of the model for a problem of a concrete segment exposed to transient heating according to three different fire scenarios. Here, the focus is on the short-term pore pressure build up, which can lead to explosive spalling of concrete and catastrophic failures of concrete structures.

1 Introduction

The hygro-thermal behavior of concrete plays a crucial role in the assessment of the reliability and lifetime of concrete structures. The impact of heat and mass transfer processes become particularly evident at high temperatures, where the increased pressure in pores, large temperature gradients and temperature induced-creep may lead to catastrophic service failures. Examples of such severe situations include hypothetical nuclear reactor accidents [30], fires in tunnels and tall buildings [31] or simulation of airfield concrete pavements [21]. Since high-temperature experiments are very expensive, predictive modeling of humidity migration, pore pressure development and damage distribution can result in significant economic savings. The first mathematical models of concrete exposed to temperatures exceeding 100°C were formulated by Bažant and Thonguthai in [5]. Since then, a considerable effort has been invested into detailed numerical simulations of concrete structures subject to high temperatures. However, much less attention has been given to the qualitative properties of the model.

Let us briefly present the single-fluid-phase, purely macroscopic, model for prediction of hygro-thermal behavior of heated concrete. The one-dimensional heat and mass transport in concrete wall is governed by the following parabolic system [4, 5, 7]:

energy conservation equation for concrete:

$$(C_w w \theta + \rho_s C_s \theta)_t = (\Lambda(\theta, P) \theta_x)_x + (C_w \kappa(\theta, P) \theta P_x)_x - h_d d_t; \quad (1)$$

water content conservation equation:

$$w_t = (\kappa(\theta, P) P_x)_x + d_t. \quad (2)$$

The primary unknowns in the model are temperature θ , water content w (the mass of all free evaporable water per m³ of concrete) and pore pressure P . In order to complete the system (1)–(2), let us note that the water content w is connected with temperature T and pore pressure P via sorption isotherms $\Phi = P/P_{sat} = \Phi(\theta, w)$ (Φ denotes the relative humidity and P_{sat} is saturation vapor pressure), which need to be determined experimentally. The thermal conductivity Λ and

permeability κ are assumed to be positive smooth functions of their arguments. Further, C_w represents the isobaric heat capacity of moisture, ρ_S and C_S , respectively, are the mass density and the isobaric heat capacity of solid microstructure (excluding hydrate water), d is the total mass of the free water released in the pores by dehydration and h_d denotes the enthalpy of dehydration per unit mass.

To complete the model, one should prescribe the appropriate boundary conditions across the boundary and initial conditions on primary unknowns. In the case of heat transfer through the boundary, the Neumann type or radiative (for heat flux) boundary conditions, respectively, are frequently under consideration in practice:

$$-\Lambda(\theta, P)\theta_x \cdot n = \alpha_c(\theta - \theta_\infty), \quad (3)$$

$$-\Lambda(\theta, P)\theta_x \cdot n = \alpha_c(\theta - \theta_\infty) + e\sigma(\theta^4 - \theta_\infty^4), \quad (4)$$

respectively, in which α_c designates the film coefficient for the heat transfer, and θ_∞ is temperature of the environment. The last expression in (4) expresses the radiative contribution to the heat flux, quantified by the Stefan-Boltzmann law in terms of the relative surface emissivity e and the Stefan-Boltzmann constant σ and the temperature difference $(\theta^4 - \theta_\infty^4)$.

The humidity flux across the boundary is quantified by the Newton law:

$$-\kappa(\theta, P)P_x \cdot n = \beta_c(P - P_\infty), \quad (5)$$

where the right hand side represents the humidity dissipated into the surrounding medium with the water vapor pressure in the environment P_∞ and β_c represents the surface emissivity of water.

The system (1)–(2), accompanied by the appropriate boundary and initial conditions, represents a challenging mathematical problem. Models of hygro-thermal transport processes in concrete are associated to systems of doubly and strongly nonlinear parabolic equations of type (written in operator form)

$$\partial_t \mathcal{B}(\mathbf{U}) - \nabla \cdot \mathcal{A}(\mathbf{U}, \nabla \mathbf{U}) = \mathcal{F}(\mathbf{U}, \nabla \mathbf{U}).$$

Although there is no general existence and regularity theory for such problems, some partial outcomes assuming special structure of operators \mathcal{A} and \mathcal{B} and growth conditions on \mathcal{F} can be found in the literature. The existence of weak solutions subject to mixed boundary conditions with homogeneous Neumann boundary conditions has been shown by Alt & Luckhaus in [3]. They obtained an existence result assuming the operator \mathcal{B} in the parabolic part to be only (weak) monotone and subgradient. This result has been extended e.g. by Filo & Kačur in [18], who proved the local existence of the weak solution for the system with nonlinear Neumann boundary conditions and under more general growth conditions on nonlinearities in \mathbf{U} . These results, however, are not applicable if \mathcal{B} does not take the subgradient form, which is typical of coupled heat and mass transport models. Thus, the analysis needs to exploit the specific structure of the problem. In this context, Vala in [33] proved the existence of solution to the purely diffusive hygro-thermal model with non-symmetry in the parabolic part but with unrealistic symmetry in the elliptic term. In [24] and [25], Li *et al.* study a coupled model for heat and mass transport arising from textile industry, which is described by a degenerate and strongly coupled parabolic system. They prove the global existence for one-dimensional problem using the Leray-Schauder fixed point theorem.

Additional results are available for the time discrete setting. Dalík *et al.* [13] analyzed the numerical solution of the Kiessl model for moisture and heat transfer in porous materials. They proved some existence and regularity results and suggested an efficient numerical approach to the solution of the resulting system of highly non-linear equations. However, the Kiessl model is valid for limited temperature range only and as such it is inappropriate for high-temperature applications. In [9], Beneš *et al.* extended the work [13] by proving the existence of an approximate solution for the Bažant–Thonguthai model, arising from the semi-implicit discretization in time.

In the present paper we prove a global-in-time existence of a weak solution to a fully nonlinear coupled parabolic system with nonlinear boundary conditions modeling heat and moisture transport in concrete walls at high temperatures based on the single-fluid-phase model introduced by Bažant & Thonguthai [5]. The main result is proved by means of a fixed point argument based on the Leray-Schauder approach and approximation procedure and then carrying out the passage to the limit.

Outline of the paper. The paper is organized in the following manner. In Section 2, we introduce basic notation and the appropriate function spaces, present the classical formulation of the problem

under consideration and specify our assumptions on data and structure conditions in the model. In Section 3, we formulate the problem in the variational sense and state the main result of the paper – the global-in-time existence of the weak solution. In Section 4, we derive the a priori estimates for an approximate solution of the auxiliary regularized problem. The solution of the regularized problem is obtained by the Leray-Schauder fixed point theorem. Using the limiting procedure we prove the existence of the weak solution to the original problem. In Section 5, numerical experiments are performed to present the moisture migration, temperature distribution and pore pressure build up in the model of concrete wall one-side-exposed to various fire scenarios.

2 Preliminaries

2.1 Notation and some function spaces

Vectors, vector functions and operators acting on vector functions are denoted by boldface letters. Throughout the paper, we will always use positive constants c, c_1, c_2, \dots , which are not specified and which may differ from line to line. As usual, for a function $\phi = \phi(x, t)$, ϕ_x and ϕ_t indicate the partial derivatives with respect to spatial variable x and temporal variable t . Let $T > 0$ and $\ell > 0$ be the fixed values, $\Omega = (0, \ell)$, $I = (0, T)$, $Q_T = \Omega \times I$, $Q_t = \Omega \times (0, t]$ ($t > 0$). We denote by $\mathbf{W}^{l,p} \equiv W^{l,p}(\Omega)^2$, $l \geq 0$ and $1 \leq p \leq \infty$ and, especially, $\mathbf{L}^p \equiv \mathbf{W}^{0,p}$, where $W^{k,p}(\Omega)$, $k \geq 0$, $p \in [1, +\infty]$, denotes the usual Sobolev space with the norm $\|\cdot\|_{W^{k,p}(\Omega)}$ and, in addition, $L^p(\Omega)$ denotes the usual Lebesgue space equipped with the norm $\|\cdot\|_{L^p(\Omega)}$.

Let X be an arbitrary Banach space with the norm $\|\cdot\|_X$ (X^* represents the dual space to X). Let $r \in [1, \infty)$. As usual $L^r(I; X)$ and $L^\infty(I; X)$ denote the Banach spaces

$$\left\{ \phi; \phi(t) \in X \text{ for almost every } t \in I, \int_0^T \|\phi(t)\|_X^r dt < \infty \right\}$$

and

$$\left\{ \phi; \phi(t) \in X \text{ for almost every } t \in I, \operatorname{ess\,sup}_{t \in I} \|\phi(t)\|_X < \infty \right\}$$

with the norms

$$\|\phi\|_{L^p(I; X)} := \left(\int_0^T \|\phi(t)\|_X^r dt \right)^{1/r}$$

and

$$\|\phi\|_{L^\infty(I; X)} := \operatorname{ess\,sup}_{t \in I} \|\phi(t)\|_X.$$

Further we introduce the space $C(I; X)$ – space of functions $\phi : [0, T] \rightarrow X$, continuous, for which

$$\|\phi\|_{C(I; X)} := \operatorname{ess\,sup}_{t \in [0, T]} \|\phi(t)\|_X.$$

Define the spaces

$$V_2^{2,1}(Q_T) \equiv \{ \phi \in L^2(Q_T) \mid \phi_t, \phi_x, \phi_{xx} \in L^2(Q_T) \} \text{ and } \mathbf{V}_2^{2,1}(Q_T) \equiv V_2^{2,1}(Q_T)^2.$$

We will often use the following well-known embeddings that are consequences of Aubin-Lions lemma and interpolation inequalities:

$$V_2^{2,1}(Q_T) \hookrightarrow L^2(I; \mathbf{W}^{1,2}), \quad (6)$$

$$L^2(I; W^{1,2}(\Omega)) \cap L^\infty(I; L^2(\Omega)) \hookrightarrow L^6(Q_T), \quad (7)$$

$$V_2^{2,1}(Q_T) \hookrightarrow C(\overline{Q_T}). \quad (8)$$

2.2 Classical formulation of the problem

Incorporating the relation $P = P_{sat}(\theta)\Phi(\theta, w) \equiv \mathcal{P}(\theta, w)$ (via sorption isotherms $\Phi = P/P_{sat} = \Phi(\theta, w)$) into the system (1)–(5) we can eliminate the unknown field P and consider the problem with only two unknowns θ and w . Consequently, the classical formulation of the problem we are

going to study reads as follows:

$$w_t - d_t = (\delta_w(\theta, w)w_x)_x + (\delta_\theta(\theta, w)\theta_x)_x \quad \text{in } Q_T, \quad (9)$$

$$(C_w w\theta + \rho_S C_S \theta)_t + h_d d_t = (\lambda(\theta, w)\theta_x)_x + C_w(\theta(\delta_w(\theta, w)w_x + \delta_\theta(\theta, w)\theta_x))_x \quad \text{in } Q_T, \quad (10)$$

$$d_t = -\frac{1}{\tau}(d - d_{eq}(\theta)) \quad \text{in } Q_T, \quad (11)$$

$$[\delta_w(\theta, w)w_x + \delta_\theta(\theta, w)\theta_x] \Big|_{x=0} = 0 \quad \text{in } I, \quad (12)$$

$$[\lambda(\theta, w)\theta_x] \Big|_{x=0} = 0 \quad \text{in } I, \quad (13)$$

$$[-\delta_w(\theta, w)w_x - \delta_\theta(\theta, w)\theta_x] \Big|_{x=\ell} = \beta_c(\mathcal{P}(\theta(\ell, t), w(\ell, t)) - P_\infty) \quad \text{in } I, \quad (14)$$

$$[-\lambda(\theta, w)\theta_x] \Big|_{x=\ell} = (\alpha_c + e\sigma|\theta(\ell, t)|^3)\theta(\ell, t) - \vartheta(t) \quad \text{in } I, \quad (15)$$

$$d(x, 0) = 0 \quad \text{in } \Omega, \quad (16)$$

$$\theta(x, 0) = \theta_0(x) \quad \text{in } \Omega, \quad (17)$$

$$w(x, 0) = w_0(x) \quad \text{in } \Omega. \quad (18)$$

Here we assume that all functions are smooth enough. The unknowns in the proposed model are temperature θ , water content w and the function d (the total mass of the free water released in the pores by dehydration). λ , δ_θ and δ_w are diffusion coefficient functions depending non-linearly on θ and w . In (11) τ represents the characteristic time of dehydration and the function $d_{eq} = d_{eq}(\theta)$ defines the water mass created by dehydration at equilibrium at the given temperature θ [12]. In (15) the function ϑ expresses thermal loading given by (cf. (4))

$$\vartheta(t) := \alpha_c \theta_\infty(t) + e\sigma \theta_\infty^4(t). \quad (19)$$

Finally, θ_0 and w_0 represent the initial distributions of the primary unknowns θ and w , respectively.

2.3 Structural conditions and assumptions on physical parameters

Here we specify assumptions on material coefficients and data in the model (9)–(18).

A_1 We assume that the material parameters ρ_S , C_S , C_w , h_d , τ , α_c , β_c , P_∞ , σ and e are real positive constants.

A_2 δ_θ , δ_w and λ are C^1 functions, $\delta_\theta, \delta_w, \lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$, satisfying for certain positive constants δ_0 , δ_1 , δ_2 , λ_0 , λ_1 and λ_2 and for all $\mathbf{z} = [z_1, z_2] \in \mathbb{R}^2$

$$0 < \delta_0 \leq \delta_w(\mathbf{z}) \leq \delta_1 < +\infty, \quad (20)$$

$$|\partial_{z_1} \delta_w(\mathbf{z})|, |\partial_{z_2} \delta_w(\mathbf{z})| \leq \delta_2 < +\infty, \quad (21)$$

$$0 < \delta_0 \leq \delta_\theta(\mathbf{z}) \leq \delta_1 < +\infty, \quad (22)$$

$$|\delta_\theta(\mathbf{z})| \leq c|z_2|, \quad (23)$$

$$|\partial_{z_1} \delta_\theta(\mathbf{z})|, |\partial_{z_2} \delta_\theta(\mathbf{z})| \leq \delta_2 < +\infty, \quad (24)$$

$$0 < \lambda_0 \leq \lambda(\mathbf{z}) \leq \lambda_1 < +\infty, \quad (25)$$

$$|\partial_{z_1} \lambda(\mathbf{z})|, |\partial_{z_2} \lambda(\mathbf{z})| \leq \lambda_2 < +\infty. \quad (26)$$

A_3 $d_{eq} : \mathbb{R} \rightarrow \mathbb{R}$ is a function of class C^1 and there exist positive constants d_1 and d_2 such that for every $z \in \mathbb{R}$

$$0 \leq d_{eq}(z) \leq d_1 < +\infty, \quad (27)$$

$$|d'_{eq}(z)| \leq d_2 < +\infty. \quad (28)$$

A_4 We assume $\mathcal{P} : \mathbb{R}^2 \rightarrow \mathbb{R}$ to be a Lipschitz continuous function in \mathbb{R}^2 such that $\mathcal{P}(\xi_1, \xi_2) \cdot \xi_2 \geq 0$ for every $[\xi_1, \xi_2] \in \mathbb{R}^2$.

A_5 ϑ is a positive continuous function in $[0, T]$.

A_6 θ_0 and w_0 are positive continuous functions in $\overline{\Omega}$.

3 Main result

The aim of this paper is to prove the existence of a weak solution to the problem described by the system (9)–(18). Now we formulate our problem in the variational sense.

Definition 1 *The triplet $[\theta, w, d] \in L^2(I; \mathbf{W}^{1,2}) \times C(I; L^\infty(\Omega))$ is called the weak solution to the system (9)–(18) iff*

$$\begin{aligned}
& - \int_{Q_T} (w - d) \phi_t + (C_w \theta w + \rho_S C_S \theta + h_d d) \psi_t dx dt \\
& + \int_{Q_T} (\delta_w(\theta, w) w_x + \delta_\theta(\theta, w) \theta_x) \phi_x dx dt + \int_0^T \beta_c(\mathcal{P}(\theta(\ell, t), w(\ell, t)) - P_\infty) \phi(\ell, t) dt \\
& + \int_{Q_T} \lambda(\theta, w) \theta_x \psi_x dx dt - \int_{Q_T} \theta (\delta_w(\theta, w) w_x + \delta_\theta(\theta, w) \theta_x) \psi_x dx dt \\
& + \int_0^T [(\alpha_c + e\sigma|\theta(\ell, t)|^3)\theta(\ell, t) - \vartheta(t) + C_w \theta(\ell, t) \beta_c(\mathcal{P}(\theta(\ell, t), w(\ell, t)) - P_\infty)] \psi(\ell, t) dt \\
& = \int_\Omega (w_0 - d_0) \phi(x, 0) + (C_w \theta_0 w_0 + \rho_S C_S \theta_0 + h_d d_0) \psi(x, 0) dx
\end{aligned} \tag{29}$$

holds for all test functions $[\phi, \psi] \in C^\infty(\overline{Q_T})$, $\phi(x, T) = \psi(x, T) = 0 \ \forall x \in \Omega$. Here d satisfies the equation

$$d(x, t) = \frac{1}{\tau} \int_0^t e^{(s-t)/\tau} d_{eq}(\theta(x, s)) ds \tag{30}$$

for every $t \in (0, T)$ and almost every $x \in \Omega$.

Remark 2 *The coupled problem described by the system (9)–(18), being intensively used in practise, does not seem analytically studied so far. To the best of our knowledge, there is no existence result for the presented model available.*

The main result of this paper reads:

Theorem 3 (Main result) *Let the assumptions A_1 – A_6 be satisfied. Then there exists at least one weak solution $[\theta, w, d] \in L^2(I; \mathbf{W}^{1,2}) \times C(I; L^\infty(\Omega))$ of the system (9)–(18) in the sense of Definition 1.*

4 Proof of the main result

4.1 The regularized problem

In the proof of the main result we use the method of mollification in order to get the approximate solution of the problem (9)–(18). Let us introduce the mollified system

$$w_t - (\delta_w(\theta^\varepsilon, w^\varepsilon) w_x)_x - (\delta_\theta(\theta^\varepsilon, w) \theta_x^\varepsilon)_x = d_t \quad \text{in } Q_T, \tag{31}$$

$$\begin{aligned}
(C_w w \theta + \rho_S C_S \theta)_t + h_d d_t &= (\lambda(\theta^\varepsilon, w^\varepsilon) \theta_x)_x \\
&+ C_w (\theta (\delta_w(\theta^\varepsilon, w^\varepsilon) w_x + \delta_\theta(\theta^\varepsilon, w) \theta_x^\varepsilon))_x \quad \text{in } Q_T, \tag{32}
\end{aligned}$$

$$d_t = -\frac{1}{\tau} (d - d_{eq}(\theta)) \quad \text{in } Q_T. \tag{33}$$

Here $\theta^\varepsilon := J_\varepsilon \star \mathcal{E}(\theta)$ is a regularization of θ , ε is a small positive real number, J_ε denotes the standard mollifier defined for the function θ (see [1, Paragraph 2.28 and Theorem 2.29]) and $\mathcal{E}(\theta)$ is the extension operator extending θ to be zero on $\mathbb{R}^2 \setminus Q_T$ (see [1, Paragraph 5.17]). Further we set $w^\varepsilon := J_\varepsilon \star w$ and in a similar way we introduce θ_0^ε , w_0^ε and ϑ^ε , respectively, as regularizations of θ_0 , w_0 and ϑ , respectively.

Substituting (31) into (32), equations (31)–(33) can be rewritten as

$$w_t - (\delta_w(\theta^\varepsilon, w^\varepsilon) w_x)_x - (\delta_\theta(\theta^\varepsilon, w) \theta_x^\varepsilon)_x = d_t \quad \text{in } Q_T, \tag{34}$$

$$\begin{aligned}
(C_w w + \rho_S C_S \theta)_t + (C_w \theta + h_d d)_t &= (\lambda(\theta^\varepsilon, w^\varepsilon) \theta_x)_x \\
&+ C_w (\delta_w(\theta^\varepsilon, w^\varepsilon) w_x + \delta_\theta(\theta^\varepsilon, w) \theta_x^\varepsilon) \theta_x \quad \text{in } Q_T, \tag{35}
\end{aligned}$$

$$d_t = -\frac{1}{\tau} (d - d_{eq}(\theta)) \quad \text{in } Q_T. \tag{36}$$

The governing equations are supplemented by the following regularized boundary and initial conditions

$$[\delta_w(\theta^\varepsilon, w^\varepsilon)w_x + \delta_\theta(\theta^\varepsilon, w)\theta_x^\varepsilon]_{x=0} = 0 \quad \text{in } I, \quad (37)$$

$$[\lambda(\theta^\varepsilon, w^\varepsilon)\theta_x]_{x=0} = 0 \quad \text{in } I, \quad (38)$$

$$[-\delta_w(\theta^\varepsilon, w^\varepsilon)w_x - \delta_\theta(\theta^\varepsilon, w)\theta_x^\varepsilon]_{x=\ell} = \beta_c(\mathcal{P}(\theta^\varepsilon(\ell, t), w(\ell, t)) - P_\infty) \quad \text{in } I, \quad (39)$$

$$[-\lambda(\theta^\varepsilon, w^\varepsilon)\theta_x]_{x=\ell} = (\alpha_c + e\sigma|\theta^\varepsilon(\ell, t)|^3)\theta(\ell, t) - \vartheta^\varepsilon(t) \quad \text{in } I, \quad (40)$$

$$d(x, 0) = 0 \quad \text{in } \Omega, \quad (41)$$

$$\theta(x, 0) = \theta_0^\varepsilon(x) \quad \text{in } \Omega, \quad (42)$$

$$w(x, 0) = w_0^\varepsilon(x) \quad \text{in } \Omega. \quad (43)$$

In this subsection we prove the existence of a strong solution to the regularized problem (34)–(43).

Theorem 4 *The system (34)–(43) has a strong solution $[\theta_\varepsilon, w_\varepsilon] \in \mathbf{V}_2^{2,1}(Q_T)$, such that*

$$\|[\theta_\varepsilon, w_\varepsilon]\|_{L^2(I; \mathbf{W}^{1,2})} \leq c, \quad (44)$$

$$\|[\theta_\varepsilon, w_\varepsilon]\|_{L^\infty(I; \mathbf{L}^2)} \leq c, \quad (45)$$

where the constant c does not depend on ε .

Proof 4.1.0.1 *Proof of Theorem 4 follows from the a priori estimates in the “weak” classes of functions by the usual Leray-Schauder fixed point arguments.*

Solution to an auxiliary problem. *For any given couple $[\tilde{\theta}, \tilde{w}] \in L^2(I; \mathbf{W}^{1,2})$ and $0 \leq \zeta \leq 1$ consider the initial-boundary value problem*

$$w_t - \left(\delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)w_x \right)_x - \left(\delta_\theta(\tilde{\theta}^\varepsilon, w)\tilde{\theta}_x^\varepsilon \right)_x = d_t \quad \text{in } Q_T, \quad (46)$$

$$(C_w w + \rho_S C_S)\theta_t + (C_w \theta + h_d)d_t = (\lambda(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)\theta_x)_x + C_w \left(\delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)w_x + \delta_\theta(\tilde{\theta}^\varepsilon, w)\tilde{\theta}_x^\varepsilon \right) \theta_x \quad \text{in } Q_T, \quad (47)$$

$$d_t = -\frac{1}{\tau}(d - \zeta d_{eq}(\tilde{\theta})) \quad \text{in } Q_T, \quad (48)$$

$$[\delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)w_x + \delta_\theta(\tilde{\theta}^\varepsilon, w)\tilde{\theta}_x^\varepsilon]_{x=0} = 0 \quad \text{in } I, \quad (49)$$

$$[\lambda(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)\theta_x]_{x=0} = 0 \quad \text{in } I, \quad (50)$$

$$[-\delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)w_x - \delta_\theta(\tilde{\theta}^\varepsilon, w)\tilde{\theta}_x^\varepsilon]_{x=\ell} = \beta_c(\mathcal{P}(\tilde{\theta}^\varepsilon(\ell, t), w(\ell, t)) - \zeta P_\infty) \quad \text{in } I, \quad (51)$$

$$[-\lambda(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)\theta_x]_{x=\ell} = (\alpha_c + e\sigma|\tilde{\theta}^\varepsilon(\ell, t)|^3)\theta(\ell, t) - \zeta\vartheta^\varepsilon(t) \quad \text{in } I, \quad (52)$$

$$d(x, 0) = 0 \quad \text{in } \Omega, \quad (53)$$

$$\theta(x, 0) = \zeta\theta_0^\varepsilon(x) \quad \text{in } \Omega, \quad (54)$$

$$w(x, 0) = \zeta w_0^\varepsilon(x) \quad \text{in } \Omega. \quad (55)$$

The proof of the existence of the solution to the system (46)–(55) is split into three steps:

Step 1 *First we treat the initial problem (48) and (53). Let d be a solution of the ordinary differential equation*

$$d_t + \frac{1}{\tau}d = \frac{1}{\tau}\zeta d_{eq}(\tilde{\theta})$$

(which holds for almost every $t \in (0, T)$ and $x \in \Omega$) with the initial condition

$$d(x, 0) = 0 \quad \text{in } \Omega.$$

Then

$$d(x, t) = \frac{1}{\tau} \int_0^t e^{(s-t)/\tau} \zeta d_{eq}(\tilde{\theta}(x, s)) ds \quad (56)$$

for every $t \in (0, T)$ and almost every $x \in \Omega$. Hence, for almost every $x \in \Omega$ we have $d(x, \cdot) \in W^{1,\infty}(I)$. Moreover, d and d_t are nonnegative almost everywhere in Q_T ($d(x, \cdot)$ is nondecreasing with respect to t).

Step 2 The system of equations (46), (49), (51) and (55) (written in non-divergence form) is a special case of the problem (cf. [23, Chapter V., (7.1)–(7.2)])

$$\begin{aligned} w_t - a(x, t)w_{xx} - b(x, t, w, w_x) &= 0 && \text{in } Q_T, \\ a(0, t)w_x(0, t) + g_1(0, t, w) &= 0 && \text{in } I, \\ -a(\ell, t)w_x(\ell, t) + g_2(\ell, t, w) &= 0 && \text{in } I, \\ w(x, 0) &= \zeta w_0^\varepsilon(x) && \text{in } \Omega. \end{aligned}$$

In our case we have

$$\begin{aligned} a(x, t) &= \delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon), \\ b(x, t, w, w_x) &= \left(\delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon) \right)_x w_x + \left(\delta_\theta(\tilde{\theta}^\varepsilon, w) \right)_x \tilde{\theta}_x^\varepsilon + \delta_\theta(\tilde{\theta}^\varepsilon, w) \tilde{\theta}_{xx}^\varepsilon + d_t, \\ g_1(0, t, w) &= \delta_\theta(\tilde{\theta}^\varepsilon(0, t), w(0, t)) \tilde{\theta}_x^\varepsilon(0, t), \\ g_2(\ell, t, w) &= -\delta_\theta(\tilde{\theta}^\varepsilon(\ell, t), w(\ell, t)) \tilde{\theta}_x^\varepsilon(\ell, t) - \beta_c \mathcal{P}(\tilde{\theta}^\varepsilon(\ell, t), w(\ell, t)) + \beta_c \zeta P_\infty. \end{aligned}$$

Under the assumptions A_1 – A_4 and A_6 , following the classical parabolic-equation theory for quasi-linear equations [23], for any given $[\tilde{\theta}, \tilde{w}] \in L^2(I; \mathbf{W}^{1,2})$ and d given by (56) the problem (46), (49), (51) and (55) admits the unique solution $w \in V_2^{2,1}(Q_T)$ (see [23, Chapter V., Theorem 7.4]). Note that $V_2^{2,1}(Q_T) \hookrightarrow C(\overline{Q}_T)$. Now we prove that w is non-negative in Q_T . Let $w^+ = \max\{0, w\}$ and $w^- = \max\{0, -w\}$. Test (46) by w^- to get

$$\begin{aligned} & \frac{1}{2} \int_\Omega |w^-(t)|^2 dx + \int_{Q_t} \delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon) |w_x^-|^2 dx ds + \int_{Q_t} d_t w^- dx ds \\ & + \int_0^t \beta_c \zeta P_\infty w^-(\ell, s) ds - \int_0^t \beta_c \mathcal{P}(\tilde{\theta}^\varepsilon(\ell, s), w(\ell, s)) w^-(\ell, s) ds \\ & = - \int_{Q_t} \delta_\theta(\tilde{\theta}^\varepsilon, w) \tilde{\theta}_x^\varepsilon w_x^- dx ds. \end{aligned} \quad (57)$$

Since $w = w^+ - w^-$ we can write

$$\begin{aligned} \int_0^t \mathcal{P}(\tilde{\theta}^\varepsilon(\ell, s), w(\ell, s)) w^-(\ell, s) ds &= \int_0^t \mathcal{P}(\tilde{\theta}^\varepsilon(\ell, s), -w^-(\ell, s)) w^-(\ell, s) ds \\ &= - \int_0^t \mathcal{P}(\tilde{\theta}^\varepsilon(\ell, s), -w^-(\ell, s)) (-w^-(\ell, s)) ds \end{aligned}$$

and due to A_4 the last integral on the left-hand side of the equation (57) is always non-positive. Hence, applying (23) and the Young's inequality to the term on the right hand side and neglecting the non-negative terms on the left-hand side, we arrive at the inequality

$$\begin{aligned} \frac{1}{2} \int_\Omega |w^-(t)|^2 dx + \int_{Q_t} \delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon) |w_x^-|^2 dx ds &\leq \int_{Q_t} c |w^-| |\tilde{\theta}_x^\varepsilon| |w_x^-| dx ds \\ &\leq c(\xi) \|\tilde{\theta}_x^\varepsilon\|_{L^\infty(Q_t)} \int_{Q_t} |w^-|^2 dx ds \\ &\quad + \xi \|\tilde{\theta}_x^\varepsilon\|_{L^\infty(Q_t)} \int_{Q_t} |w_x^-|^2 dx ds. \end{aligned}$$

Hence, choosing ξ sufficiently small and taking into account (20), we can use the Gronwall lemma to conclude $w^- \equiv 0$ and thus $w \geq 0$ in Q_T .

Step 3 Now having $w \in V_2^{2,1}(Q_T)$, $w \geq 0$, we get θ as the solution of the system (47), (50), (52) and (54). We can write this linear parabolic problem in the form

$$\theta_t - a_1(x, t)\theta_{xx} - a_2(x, t)\theta_x + a_3(x, t)\theta = f(x, t) \quad \text{in } Q_T, \quad (58)$$

$$\theta_x(0, t) = 0 \quad \text{in } I, \quad (59)$$

$$-\lambda(\tilde{\theta}^\varepsilon(\ell, t), \tilde{w}^\varepsilon(\ell, t))\theta_x(\ell, t) - (\alpha_c + e\sigma|\tilde{\theta}^\varepsilon(\ell, t)|^3)\theta(\ell, t) = \zeta\vartheta^\varepsilon(t) \quad \text{in } I, \quad (60)$$

$$\theta(x, 0) = \zeta\theta_0^\varepsilon(x) \quad \text{in } \Omega, \quad (61)$$

where

$$\begin{aligned} a_1(x, t) &= \frac{\lambda(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)}{(C_w w + \rho_S C_S)}, \\ a_2(x, t) &= \frac{1}{(C_w w + \rho_S C_S)} \left[\lambda_x(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon) + C_w \left(\delta_w(\tilde{\theta}^\varepsilon, \tilde{w}^\varepsilon)w_x + \delta_\theta(\tilde{\theta}^\varepsilon, w)\tilde{\theta}_x^\varepsilon \right) \right], \\ a_3(x, t) &= \frac{C_w d_t}{(C_w w + \rho_S C_S)}, \\ f(x, t) &= \frac{-h_d d_t}{(C_w w + \rho_S C_S)} \end{aligned}$$

and $a_1 \in C(\overline{Q_T})$, $a_2 \in L^{2+\epsilon}(I; L^\infty(\Omega))$ (ϵ is a small positive number), a_3 and $f \in L^\infty(\Omega)$. By the linear theory for parabolic problems (see [15, Theorem 2.1]) there exists the uniquely determined solution $\theta \in V_2^{2,1}(Q_T)$ of the problem (58)–(61).

Finally, let us conclude that for any given couple $[\tilde{\theta}, \tilde{w}] \in L^2(I; \mathbf{W}^{1,2})$ and $0 \leq \zeta \leq 1$ we have $[\theta, w] \in \mathbf{V}_2^{2,1}(Q_T)$ as the solution of the problem (46)–(55).

Basic a priori estimates. In this paragraph we prove some a priori estimates for θ and w .

Test (34) by $C_w \theta^2$ and (35) by 2θ to obtain (adding these both resulting equations)

$$\begin{aligned} & \int_{Q_t} (\theta^2(C_w w + \rho_S C_S))_s \, dx ds + \int_{Q_t} 2\lambda(\theta^\varepsilon, w^\varepsilon)|\theta_x|^2 \, dx ds + \int_{Q_t} C_w \theta^2 d_s \, dx ds \\ & + \int_{Q_t} 2h_d \theta d_s \, dx ds + \int_0^t C_w \theta(\ell, s)^2 \beta_c (\mathcal{P}(\theta^\varepsilon(\ell, s), w(\ell, s)) - \zeta P_\infty) \, ds \\ & + \int_0^t 2\theta(\ell, s) ((\alpha_c + e\sigma|\theta^\varepsilon(\ell, s)|^3)\theta(\ell, s) - \zeta\vartheta^\varepsilon(s)) \, ds = 0. \end{aligned} \quad (62)$$

Hence, simple modifications yield

$$\begin{aligned} & \int_\Omega \theta(x, t)^2 (C_w w(x, t) + \rho_S C_S) \, dx + \int_{Q_t} 2\lambda(\theta^\varepsilon, w^\varepsilon)|\theta_x|^2 \, dx ds + \int_{Q_t} C_w \theta^2 d_s \, dx ds \\ & + \int_0^t C_w \theta(\ell, s)^2 \beta_c \mathcal{P}(\theta^\varepsilon(\ell, s), w(\ell, s)) \, ds + \int_0^t 2\theta^2(\ell, s)(\alpha_c + e\sigma|\theta^\varepsilon(\ell, s)|^3) \, ds \\ & = \int_\Omega \theta(x, 0)^2 (C_w w(x, 0) + \rho_S C_S) \, dx - \int_{Q_t} 2h_d \theta d_s \, dx ds \\ & + \zeta \int_0^t C_w \theta(\ell, s)^2 \beta_c P_\infty \, ds + \zeta \int_0^t 2\theta(\ell, s)\vartheta^\varepsilon(s) \, ds. \end{aligned} \quad (63)$$

The last three integrals can be estimated using Young's inequality and [18, Remark 4], respectively, as follows:

$$\int_{Q_t} 2h_d \theta d_s \, dx ds \leq h_d \int_{Q_t} \theta^2 \, dx ds + h_d \int_{Q_t} d_s^2(x, s) \, dx ds, \quad (64)$$

$$\int_0^t C_w \theta(\ell, s)^2 \beta_c P_\infty \, ds \leq C_w \beta_c P_\infty \int_{Q_t} \epsilon |\theta_x|^2 + C(\epsilon) |\theta|^2 \, dx ds \quad (65)$$

and

$$\int_0^t 2\theta(\ell, s)\vartheta^\varepsilon(s) \, ds \leq \epsilon \int_0^t \theta^2(\ell, s) \, ds + C(\epsilon) \int_0^t \vartheta^\varepsilon(s)^2 \, ds, \quad (66)$$

where ϵ represents sufficiently small positive real number. Note that since $w \geq 0$, A_4 yields

$$\int_0^t C_w \theta(\ell, s)^2 \beta_c \mathcal{P}(\theta^\epsilon(\ell, s), w(\ell, s)) ds \geq 0.$$

Taking into account (25), (64)–(66) and neglecting the non-negative terms on the left-hand side in (63) we arrive at the estimate

$$c_1 \int_{\Omega} \theta(x, t)^2 dx + c_2 \int_{Q_t} |\theta_x|^2 dx ds + c_3 \int_0^t \theta^2(\ell, s) ds \leq c_4 + c_5 \int_{Q_t} \theta^2 dx ds. \quad (67)$$

Hence, by means of Gronwall's lemma one checks that

$$\|\theta\|_{L^2(I; W^{1,2}(\Omega))} \leq c, \quad (68)$$

$$\|\theta\|_{L^\infty(I; L^2(\Omega))} \leq c \quad (69)$$

for the positive constant c being independent of ζ and ϵ .

Now let us derive some uniform estimates for water content w . Multiplying the equation (34) by w and integrating over Q_t we get

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} w(x, t)^2 dx + \int_{Q_t} \delta_w(\theta^\epsilon, w^\epsilon) |w_x|^2 dx ds + \int_{Q_t} \delta_\theta(\theta^\epsilon, w) \theta_x^\epsilon w_x dx ds \\ & \quad + \int_0^t w(\ell, s) \beta_c (\mathcal{P}(\theta^\epsilon(\ell, s), w(\ell, s)) - \zeta P_\infty) ds \\ & = \frac{1}{2} \int_{\Omega} w(x, 0)^2 dx + \int_{Q_t} w d_s dx ds \end{aligned} \quad (70)$$

and consequently

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} w(x, t)^2 dx + \int_{Q_t} \delta_w(\theta^\epsilon, w^\epsilon) |w_x|^2 dx ds + \int_0^t w(\ell, s) \beta_c \mathcal{P}(\theta^\epsilon(\ell, s), w(\ell, s)) ds \\ & \leq \frac{1}{2} \int_{\Omega} w(x, 0)^2 dx + \frac{1}{2} \int_{Q_t} w^2 dx ds + \frac{1}{2} \int_{Q_t} d_s^2(x, s) dx ds \\ & \quad - \int_{Q_t} \delta_\theta(\theta^\epsilon, w) \theta_x^\epsilon w_x dx ds + \zeta \int_0^t w(\ell, s) \beta_c P_\infty ds. \end{aligned} \quad (71)$$

By A_4 we deduce

$$\int_0^t w(\ell, s) \beta_c \mathcal{P}(\theta^\epsilon(\ell, s), w(\ell, s)) ds \geq 0. \quad (72)$$

Further, by (22) and the Young's inequality we have

$$\begin{aligned} \int_{Q_t} \delta_\theta(\theta^\epsilon, w) \theta_x^\epsilon w_x dx ds & \leq \epsilon \int_{Q_t} w_x^2 dx ds + C(\epsilon) \int_{Q_t} |\delta_\theta(\theta^\epsilon, w)|^2 |\theta_x^\epsilon|^2 dx ds \\ & \leq \epsilon \int_{Q_t} w_x^2 dx ds + c_1 C(\epsilon) \|\theta\|_{L^2(I; W^{1,2}(\Omega))}^2, \end{aligned} \quad (73)$$

while using the Cauchy's inequality and [18, Remark 4] one obtains the estimate

$$\begin{aligned} \int_0^t w(\ell, s) \beta_c P_\infty ds & \leq c_1 + c_2 \int_0^t w^2(\ell, s) ds \\ & \leq c_1 + c_2 \int_{Q_t} \epsilon w_x^2 + C(\epsilon) w^2 dx ds. \end{aligned} \quad (74)$$

Now taking (71)–(74) together and using (20) and (68), we arrive at the estimate of the form

$$c_1 \int_{\Omega} w(x, t)^2 dx + c_2 \int_{Q_t} w_x^2 dx ds + c_3 \int_{Q_t} w^2 dx ds \leq c_4 + c_5 \int_{Q_t} w^2 dx ds. \quad (75)$$

By the Gronwall's inequality we get the uniform estimates for w

$$\|w\|_{L^2(I; W^{1,2}(\Omega))} \leq c, \quad (76)$$

$$\|w\|_{L^\infty(I; L^2(\Omega))} \leq c \quad (77)$$

for the positive constant c being independent of ζ and ϵ .

Having established the a priori estimates for the solution of the regularized problem we are ready to complete the proof of Theorem 4 using the Leray–Schauder approach.

Leray-Schauder fixed point arguments. Denote by $X = L^2(I; \mathbf{W}^{1,2})$, take arbitrary $[\tilde{\theta}, \tilde{w}] \in X$, $\zeta \in [0, 1]$ and define the couple $[\theta, w] \in \mathbf{V}_2^{2,1}(Q_T)$ as the solution of the problem (46)–(55). Define the nonlinear mapping $\mathcal{A} : X \times [0, 1] \rightarrow X$, given by the equation $[\theta, w] = \mathcal{A}([\tilde{\theta}, \tilde{w}, \zeta])$. It is a technical procedure to check that the mapping \mathcal{A} is continuous and compact (cf. (6)). For $\zeta = 0$ we have $\mathcal{A}([\tilde{\theta}, \tilde{w}, 0]) = [0, 0]$ for all $[\tilde{\theta}, \tilde{w}] \in X$. The estimates (68) and (76) imply that $[\theta, w]$, the solution of the problem $[\theta, w] = \mathcal{A}([\theta, w, \zeta])$ for some $\zeta \in [0, 1]$, is uniformly bounded in X . Now the existence of at least one fixed point $[\theta_\varepsilon, w_\varepsilon] \in X$, $\mathcal{A}[\theta_\varepsilon, w_\varepsilon, 1] = [\theta_\varepsilon, w_\varepsilon]$, follows from the Leray-Schauder theorem [26]. Consequently, $[\theta_\varepsilon, w_\varepsilon] = \mathcal{A}([\theta_\varepsilon, w_\varepsilon, 1]) \in \mathbf{V}_2^{2,1}(Q_T)$ and $[\theta_\varepsilon, w_\varepsilon]$ is the strong solution of the system (34)–(43). The estimates (68) and (76) imply (44). Clearly, (69) and (77) yield (45). The proof of Theorem 4 is complete.

4.2 Passage to the limit for $\varepsilon \rightarrow 0$

To complete the proof of the main result stated in Theorem 3 we pass to the limit for $\varepsilon \rightarrow 0$ and study the convergence of the solution $[\theta_\varepsilon, w_\varepsilon]$ of the system (34)–(43). We present various convergence results based on the uniform estimates (68)–(69) and (76)–(77) and, in addition, prove some a priori estimates for the strong solution $[\theta_\varepsilon, w_\varepsilon]$ of the problem (34)–(43), which is equivalent to the system (31)–(33) with the boundary and initial conditions (37)–(43).

First, recall that by Theorem 4 we have

$$\|\theta_\varepsilon\|_{L^2(I; W^{1,2}(\Omega))}, \|\theta_\varepsilon\|_{L^\infty(I; L^2(\Omega))} \leq c, \quad (78)$$

$$\|w_\varepsilon\|_{L^2(I; W^{1,2}(\Omega))}, \|w_\varepsilon\|_{L^\infty(I; L^2(\Omega))} \leq c \quad (79)$$

and from the equation (46) we derive that

$$\|(w_\varepsilon)_t\|_{L^2(I; W^{1,2}(\Omega)^*)} \leq c. \quad (80)$$

As a consequence of the preceding a priori estimates we see that there exist functions $w \in L^2(I; W^{1,2}(\Omega))$ and $\theta \in L^2(I; W^{1,2}(\Omega))$, $w_t \in L^2(I; W^{1,2}(\Omega)^*)$, such that, along a selected subsequence, we have ($\varepsilon_j \rightarrow 0^+$ as $j \rightarrow \infty$)

$$w_{\varepsilon_j} \rightarrow w \quad \text{weakly in } L^2(I; W^{1,2}(\Omega)), \quad (81)$$

$$(w_{\varepsilon_j})_t \rightarrow w_t \quad \text{weakly in } L^2(I; W^{1,2}(\Omega)^*), \quad (82)$$

$$w_{\varepsilon_j} \rightarrow w \quad \text{almost everywhere in } Q_T \quad (83)$$

and

$$\theta_{\varepsilon_j} \rightarrow \theta \quad \text{weakly in } L^2(I; W^{1,2}(\Omega)). \quad (84)$$

By the embedding $L^2(I; W^{1,2}(\Omega)) \cap L^\infty(I; L^2(\Omega)) \hookrightarrow L^6(Q_T)$ we get the uniform bound (using (78) and (79))

$$\begin{aligned} & \|\theta_\varepsilon(\delta_w(\theta_\varepsilon^\varepsilon, w_\varepsilon^\varepsilon)(w_\varepsilon)_x + \delta_\theta(\theta_\varepsilon^\varepsilon, w_\varepsilon)(\theta_\varepsilon^\varepsilon)_x)\|_{L^{3/2}(Q_T)} \\ & \leq c_1 \|\theta_\varepsilon\|_{L^6(Q_T)} (\|(w_\varepsilon)_x\|_{L^2(Q_T)} + \|(\theta_\varepsilon)_x\|_{L^2(Q_T)}) \leq c_2 \end{aligned} \quad (85)$$

and similarly

$$\|C_w w_\varepsilon \theta_\varepsilon + \rho_S C_S \theta_\varepsilon\|_{L^{3/2}(0,T; W^{1,3/2}(\Omega))} \leq c. \quad (86)$$

From the equation (32) we obtain the uniform estimate

$$\|(C_w w_\varepsilon \theta_\varepsilon + \rho_S C_S \theta_\varepsilon)_t\|_{L^{3/2}(I; W^{1,3}(\Omega)^*)} \leq c. \quad (87)$$

Since

$$W^{1,3/2}(\Omega) \hookrightarrow W^{1-\beta,3/2}(\Omega) \hookrightarrow W^{1,3}(\Omega)^*,$$

where β is a small positive real number, the Aubin-Lions lemma yields the existence of $\chi \in L^{3/2}(I; W^{1-\beta,3/2}(\Omega))$ such that (modulo a subsequence)

$$C_w w_{\varepsilon_j} \theta_{\varepsilon_j} + \rho_S C_S \theta_{\varepsilon_j} \rightarrow \chi \quad \text{strongly in } L^{3/2}(I; W^{1-\beta,3/2}(\Omega)). \quad (88)$$

Since (88) yields the almost everywhere convergence and w_{ε_j} converges almost everywhere to w , we conclude

$$\theta_{\varepsilon_j} \rightarrow \theta \quad \text{almost everywhere in } Q_T. \quad (89)$$

Hence, $C_w w_{\varepsilon_j} \theta_{\varepsilon_j} + \rho_S C_S \theta_{\varepsilon_j}$ converges almost everywhere to $C_w w \theta + \rho_S C_S \theta$ and $\chi = C_w w \theta + \rho_S C_S \theta$. Finally, (87) yields

$$(C_w w_{\varepsilon_j} \theta_{\varepsilon_j} + \rho_S C_S \theta_{\varepsilon_j})_t \rightarrow (C_w w \theta + \rho_S C_S \theta)_t \quad \text{weakly in } L^{3/2}(I; W^{1,3}(\Omega)^*). \quad (90)$$

By (81)–(84) and (89) we conclude

$$\lambda(\theta_{\varepsilon_j}^\varepsilon, w_{\varepsilon_j}^\varepsilon)(\theta_{\varepsilon_j})_x \rightarrow \lambda(\theta, w)\theta_x \quad \text{weakly in } L^2(Q_T), \quad (91)$$

$$\delta_w(\theta_{\varepsilon_j}^\varepsilon, w_{\varepsilon_j}^\varepsilon)(w_{\varepsilon_j})_x + \delta_\theta(\theta_{\varepsilon_j}^\varepsilon, w_{\varepsilon_j}^\varepsilon)(\theta_{\varepsilon_j})_x \rightarrow \delta_w(\theta, w)w_x + \delta_\theta(\theta, w)\theta_x \quad \text{weakly in } L^2(Q_T) \quad (92)$$

and finally,

$$\theta_{\varepsilon_j} \left(\delta_w(\theta_{\varepsilon_j}^\varepsilon, w_{\varepsilon_j}^\varepsilon)(w_{\varepsilon_j})_x + \delta_\theta(\theta_{\varepsilon_j}^\varepsilon, w_{\varepsilon_j}^\varepsilon)(\theta_{\varepsilon_j})_x \right) \rightarrow \theta (\delta_w(\theta, w)w_x + \delta_\theta(\theta, w)\theta_x) \quad \text{weakly in } L^{3/2}(Q_T). \quad (93)$$

Now let us present the convergence of the boundary conditions. By (76) and (82) we deduce that there exists a subsequence (not relabeled), such that

$$w_{\varepsilon_j}(0, \cdot) \rightarrow w(0, \cdot) \quad \text{and} \quad w_{\varepsilon_j}(\ell, \cdot) \rightarrow w(\ell, \cdot) \quad \text{strongly in } L^2(I) \text{ and almost everywhere in } I. \quad (94)$$

Taking fixed $x = 0$ or $x = \ell$ and using (88) with $\chi = C_w w \theta + \rho_S C_S \theta$ we conclude

$$C_w w_{\varepsilon_j}(0, \cdot) \theta_{\varepsilon_j}(0, \cdot) + \rho_S C_S \theta_{\varepsilon_j}(0, \cdot) \rightarrow C_w w(0, \cdot) \theta(0, \cdot) + \rho_S C_S \theta(0, \cdot) \quad \text{almost everywhere in } I, \quad (95)$$

$$C_w w_{\varepsilon_j}(\ell, \cdot) \theta_{\varepsilon_j}(\ell, \cdot) + \rho_S C_S \theta_{\varepsilon_j}(\ell, \cdot) \rightarrow C_w w(\ell, \cdot) \theta(\ell, \cdot) + \rho_S C_S \theta(\ell, \cdot) \quad \text{almost everywhere in } I. \quad (96)$$

Now (94)–(96) imply

$$\theta_{\varepsilon_j}(0, \cdot) \rightarrow \theta(0, \cdot) \quad \text{and} \quad \theta_{\varepsilon_j}(\ell, \cdot) \rightarrow \theta(\ell, \cdot) \quad \text{almost everywhere in } I. \quad (97)$$

In order to get convergence results applicable to the radiative boundary conditions, we need “better” uniform estimates than (78).

Lemma 5 *Let $[\theta_\varepsilon, w_\varepsilon] \in \mathbf{V}_2^{2,1}(Q_T)$ be the solution of the system (34)–(43). Then θ_ε is uniformly bounded in $L^\infty(I; L^4(\Omega))$, i.e.*

$$\|\theta_\varepsilon\|_{L^\infty(I; L^4(\Omega))} \leq c, \quad (98)$$

where the constant c does not depend on ε .

Proof 4.2.0.2 *Test (34) by $C_w \theta^4$ and (35) by $4\theta^3$ to obtain (adding the both resulting equations)*

$$\begin{aligned} & \int_\Omega \theta(x, t)^4 (C_w w(x, t) + \rho_S C_S) dx + \int_{Q_t} 12\theta^2 \lambda(\theta^\varepsilon, w^\varepsilon) \theta_x^2 dx ds + \int_{Q_t} 3C_w \theta^4 d_s dx ds \\ & + \int_{Q_t} 4h_d \theta^3 d_s dx ds + \int_0^t C_w \theta(\ell, s)^4 \beta_c (\mathcal{P}(\theta^\varepsilon(\ell, s), w(\ell, s)) - \zeta P_\infty) ds \\ & + \int_0^t 4\theta(\ell, s)^3 ((\alpha_c + e\sigma|\theta^\varepsilon(\ell, s)|^3)\theta(\ell, s) - \zeta \vartheta^\varepsilon(s)) ds \\ & = \int_\Omega \theta(x, 0)^4 (C_w w(x, 0) + \rho_S C_S) dx. \end{aligned} \quad (99)$$

Simple calculation directly leads to the inequality

$$\begin{aligned} & c_1 \int_\Omega \theta(x, t)^4 dx + c_2 \int_{Q_t} \theta^2 \theta_x^2 dx ds + c_3 \int_0^t \theta(\ell, s)^4 ds \\ & \leq \int_\Omega \theta(x, 0)^4 (C_w w(x, 0) + \rho_S C_S) dx - \int_{Q_t} 4h_d \theta^3 d_s dx ds \\ & \quad + \int_0^t C_w \theta(\ell, s)^4 \beta_c P_\infty ds + \int_0^t 4\theta(\ell, s)^3 \vartheta^\varepsilon(s) ds. \end{aligned} \quad (100)$$

The integrals on the right hand side of (100) can be estimated using Young's inequality and [18, Remark 4], respectively, in the following manner:

$$\int_{Q_t} 4h_d \theta^3 d_s dx ds \leq 3h_d \int_{Q_t} \theta^4 dx ds + h_d \int_{Q_t} d_s^4(x, s) dx ds, \quad (101)$$

further,

$$\int_0^t C_w \theta(\ell, s)^4 \beta_c P_\infty ds \leq C_w \beta_c P_\infty \left(\int_{Q_t} \epsilon 4\theta^2 \theta_x^2 + C(\epsilon) \theta^4 dx ds \right) \quad (102)$$

and finally,

$$\int_0^t 4\theta(\ell, s)^3 \vartheta^\varepsilon(s) ds \leq \epsilon \int_0^t \theta^4(\ell, s) ds + C(\epsilon) \int_0^t \vartheta^\varepsilon(s)^4 ds, \quad (103)$$

where ϵ represents sufficiently small positive real number. Now, taking (100)–(103) together, we obtain

$$c_1 \int_\Omega \theta(x, t)^4 dx \leq c_2 + c_3 \int_{Q_t} \theta^4 dx ds, \quad (104)$$

which yields, applying the Gronwall's inequality, the uniform estimate (98). The proof is complete.

By the Sobolev embedding theorem (see [1, Theorem 4.12] or [22, Theorem 8.1.2]) we have

$$W^{1,2}(\Omega) \hookrightarrow W^{3/4,4}(\Omega). \quad (105)$$

Raising and integrating the interpolation inequality [1, Theorem 5.2] (ϵ means small positive real number)

$$\|\theta_\varepsilon\|_{W^{1/4+\epsilon,4}(\Omega)} \leq c \|\theta_\varepsilon\|_{W^{3/4,4}(\Omega)}^{(1+4\epsilon)/3} \|\theta_\varepsilon\|_{L^4(\Omega)}^{(2-4\epsilon)/3} \quad (106)$$

from 0 to T we get

$$\begin{aligned} \left(\int_0^T \|\theta_\varepsilon\|_{W^{1/4+\epsilon,4}(\Omega)}^{6/(1+4\epsilon)} dt \right)^{(1+4\epsilon)/6} &\leq c \left(\int_0^T \|\theta_\varepsilon\|_{W^{3/4,4}(\Omega)}^2 \|\theta_\varepsilon\|_{L^4(\Omega)}^{4(1-2\epsilon)/(1+4\epsilon)} dt \right)^{(1+4\epsilon)/6} \\ &\leq c \|\theta_\varepsilon\|_{L^2(I; W^{3/4,4}(\Omega))}^{(1+4\epsilon)/3} \|\theta_\varepsilon\|_{L^\infty(I; L^4(\Omega))}^{(2-4\epsilon)/3}. \end{aligned} \quad (107)$$

Now taking into account (78), (98), (105) and (107) we arrive at the estimate

$$\|\theta_\varepsilon\|_{L^{6/(1+4\epsilon)}(I; W^{1/4+\epsilon,4}(\Omega))} \leq c. \quad (108)$$

Consequently, we get (along a selected subsequence)

$$\theta_{\varepsilon_j}(0, \cdot) \rightarrow \theta(0, \cdot) \quad \text{and} \quad \theta_{\varepsilon_j}(\ell, \cdot) \rightarrow \theta(\ell, \cdot) \quad \text{weakly in } L^p(I), \quad 1 \leq p < 6. \quad (109)$$

The strong solution $[\theta_\varepsilon, w_\varepsilon] \in \mathbf{V}_2^{2,1}(Q_T)$ of the problem (34)–(43) (ensured by Theorem 4) is a solution of the system (31)–(33) with the boundary and initial conditions (37)–(43) and satisfies the variational problem (corresponding to (31)–(33) and (37)–(43))

$$\begin{aligned} & - \int_{Q_T} (w_\varepsilon - d_\varepsilon) \phi_t + (C_w \theta_\varepsilon w_\varepsilon + \rho_S C_S \theta_\varepsilon + h_d d_\varepsilon) \psi_t dx dt \\ & + \int_{Q_T} (\delta_w(\theta_\varepsilon^\varepsilon, w_\varepsilon^\varepsilon)(w_\varepsilon)_x + \delta_\theta(\theta_\varepsilon^\varepsilon, w_\varepsilon)(\theta_\varepsilon^\varepsilon)_x) \phi_x + \lambda(\theta_\varepsilon^\varepsilon, w_\varepsilon^\varepsilon)(\theta_\varepsilon)_x \psi_x dx dt \\ & - \int_{Q_T} \theta_\varepsilon (\delta_w(\theta_\varepsilon^\varepsilon, w_\varepsilon^\varepsilon)(w_\varepsilon)_x + \delta_\theta(\theta_\varepsilon^\varepsilon, w_\varepsilon)(\theta_\varepsilon^\varepsilon)_x) \psi_x dx dt \\ & + \int_0^T \beta_c (\mathcal{P}(\theta_\varepsilon^\varepsilon(\ell, t), w_\varepsilon(\ell, t)) - P_\infty) \phi(\ell, t) dt \\ & + \int_0^T [(\alpha_c + \epsilon \sigma |\theta_\varepsilon^\varepsilon(\ell, t)|^3) \theta_\varepsilon(\ell, t) - \vartheta^\varepsilon(t)] \psi(\ell, t) dt \\ & + \int_0^T C_w \theta_\varepsilon(\ell, t) \beta_c (\mathcal{P}(\theta_\varepsilon^\varepsilon(\ell, t), w_\varepsilon(\ell, t)) - P_\infty) \psi(\ell, t) dt \\ & = \int_\Omega w_0^\varepsilon \phi(x, 0) + (C_w \theta_0^\varepsilon w_0^\varepsilon + \rho_S C_S \theta_0^\varepsilon) \psi(x, 0) dx \end{aligned} \quad (110)$$

for all test functions $[\phi, \psi] \in C^\infty(\overline{Q}_T)$, $\phi(x, T) = \psi(x, T) = 0 \ \forall x \in \Omega$.

The above established convergences (81)–(84), (90)–(94) and (109) are sufficient for taking the limit $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ in (110) (along a selected subsequence) to get the weak solution of the system (9)–(18) satisfying (29). This completes the proof of the main result stated by Theorem 3.

5 Illustration of transport processes in concrete walls at elevated temperatures

5.1 Illustrative example

We assume a concrete wall with a width of 120 mm subjected to fire on one side while on the other side, the wall is assumed to be thermal and moisture insulated, see Fig. 1. Three different fire scenarios, which represent the time dependency of the ambient temperature on the exposed side, are employed: (i) standard fire curve (ISO fire), (ii) hydrocarbon fire curve (HC fire), and (iii) parametric fire curve (PM fire), see Fig. 2. The last one is more sophisticated than the others since for the PM fire, the temperature is dependent not only on the time of fire but also on the specific parameters of a given fire compartment (in our case, the following parameters are assumed: $q_{td} = 160 \text{ MJ m}^{-2}$, $O = 0.12 \text{ m}^{1/2}$, $b = 1000 \text{ J m}^{-2} \text{ s}^{-1/2} \text{ K}^{-1}$, fire growth rate: medium, see [17]). As shown in Fig. 2, this curve also includes a decreasing branch that simulates a cooling phase of a fire. More information about the fire scenarios mentioned above can be found in [17].

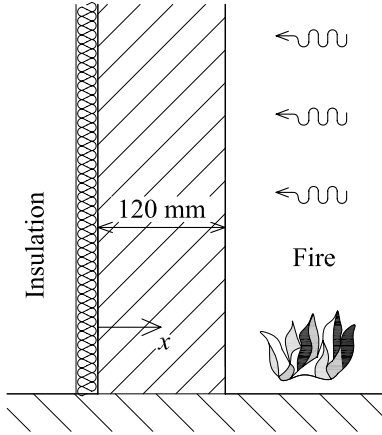


Figure 1: Analyzed wall.

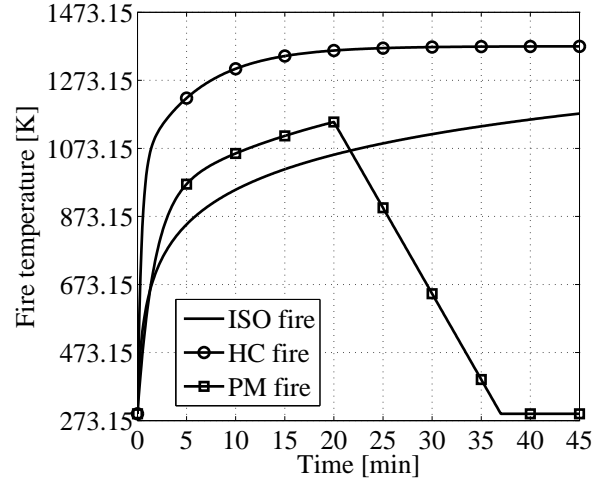


Figure 2: Fire curves.

On the exposed side of the analyzed wall, the constant parameters of the boundary conditions are set to $\beta_c = 0.019 \text{ ms}^{-1}$, $P_\infty = 1.7542 \times 10^3 \text{ Pa}$, $e = 0.7$, $\sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$, and $\alpha_c = 25, 50, \text{ or } 35 \text{ W m}^{-2} \text{ K}^{-1}$ for ISO fire, HC fire or PM fire, respectively. The temperature $\theta_\infty(t)$ is assumed to be equal to the fire temperature according to Fig. 2.

The uniform initial conditions $\theta_0 = 293.15 \text{ K}$ and $w_0 = 71.01 \text{ kg m}^{-3}$ are considered.

5.1.1 Material data

The material properties of concrete at high temperatures are assumed as follows.

The thermal conductivity of concrete $\Lambda = \Lambda(\theta, P)$ is adopted from [19, (46)–(47)], where $\Lambda_{d0} = 1.3863 \text{ W m}^{-1} \text{ K}^{-1}$, $A_\Lambda = -0.0007272 \text{ K}^{-1}$ (see [20, Tab. 2]). The concrete porosity $n = n(\theta)$ is assumed according to [32, (A7)], with $n_0 = 0.1$, and the saturation $S = S(\theta, P)$ according to [11, (10)]. The permeability of concrete $\kappa = \kappa(\theta, P) = a(\theta, P)/g$ is taken from [5, (12a)–(12b)], where $a_0 = 10^{-13} \text{ ms}^{-1}$. The mass of dehydrated water is determined by Eq. (11), which is adopted from [12, (M-4)]. Here, the term $d_{eq}(\theta)$ is taken from [2, (1.9)] with $d_{eq}^{378.15 \text{ K}} = 330 \text{ kg m}^{-3}$ and $\tau = 10800 \text{ s}$. The sorption isotherm functions are assumed according to [5] with some modification based on [14, (73)] and [16, (29)]. Here, the mass of anhydrous cement per unit volume of concrete $c = 250 \text{ kg m}^{-3}$ and the saturation water content at the room temperature $w_{0s} = 100 \text{ kg m}^{-3}$. The

parameters h_d , ρ_S , C_S and C_w are considered to be constant values, namely $h_d = 2.5 \times 10^6 \text{ J kg}^{-1}$, $\rho_S = 2400 \text{ kg m}^{-3}$, $C_S = 900 \text{ J kg}^{-1} \text{ K}^{-1}$ and $C_w = 2080 \text{ J kg}^{-1} \text{ K}^{-1}$.

5.1.2 Numerical procedure

In order to obtain an approximate solution of the nonlinear model, the well-known Galerkin procedure can be employed. The spacial discretization is performed by the one-dimensional finite element method. We consider linear elements with the element size of 0.0005 m (240 elements in total). The time discretization is carried out by a semi-implicit difference scheme. In our case, we assume the time step $\Delta t = 0.5 \text{ s}$. The numerical procedure is described in detail in [9].

An algorithm arising from the numerical scheme mentioned above has been included in an in-house MATLAB code, which is employed to determine the distribution of temperature, pore pressure and water content in the analyzed wall.

5.1.3 Results

The results obtained for two different times of fire exposure are given in Fig. 3. It is obvious that the distribution of the thermo-hygral quantities in concrete at high temperatures is dependent on the type of fire scenario used for simulation. For the nominal fires, where the fire temperature is a monotonically increasing function of time (in our case the ISO fire and HC fire curve), the temperature distribution across the analyzed wall increases with time for the whole fire exposure. Contrary to this for the PM fire, the temperature within the structure follows not only the heating phase of a fire but also the cooling period, which may lead to the decrease of temperature in some parts of a structure, see Fig. 3 and Fig. 4. This holds also for the peak values of other thermo-hygral quantities, see Fig. 3 and Fig. 4.

It is obvious, that the usage of the PM fire curve (or other natural fire scenario) provides a better prediction of a real behavior of a structure exposed to fire compared to the results obtained by assuming some of the nominal fires (e.g. ISO fire or HC fire curve). On the other hand, the nominal fire curves are still widely used due to their simplicity and also due to the fact that, in most cases, they lead to conservative results.

As shown in Figs. 3 and 4, the rapid heating of a structure and the related moisture migration induce the increase of pore pressure near the heated surface. In some cases, this pore pressure build up may lead to the spalling of concrete surface layer, and hence, to the eventual collapse of a structure, see [9].

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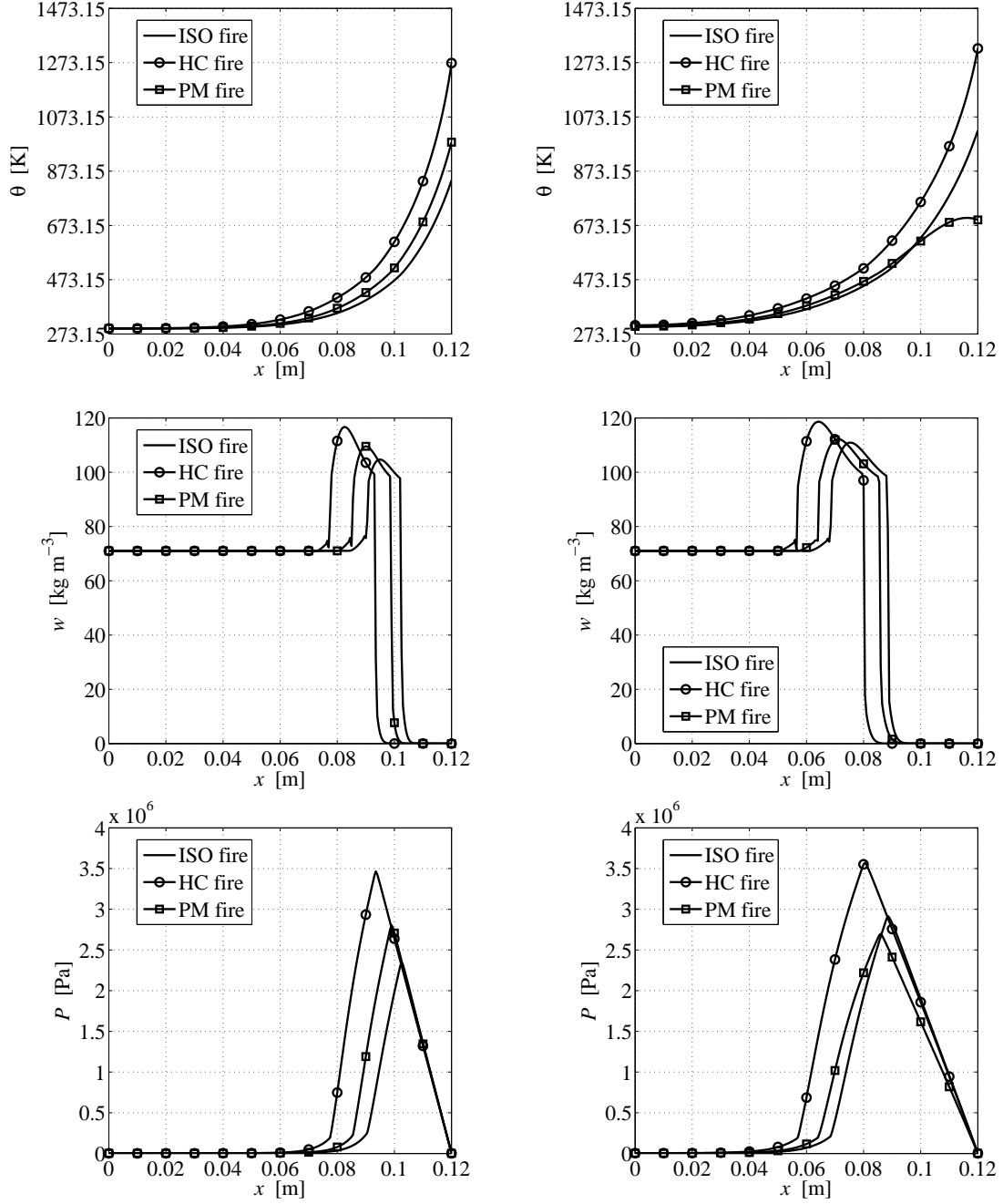


Figure 3: Temperature, water content and pore pressure distribution across the analyzed wall: fire exposure of 15 minutes (left) and 30 minutes (right).

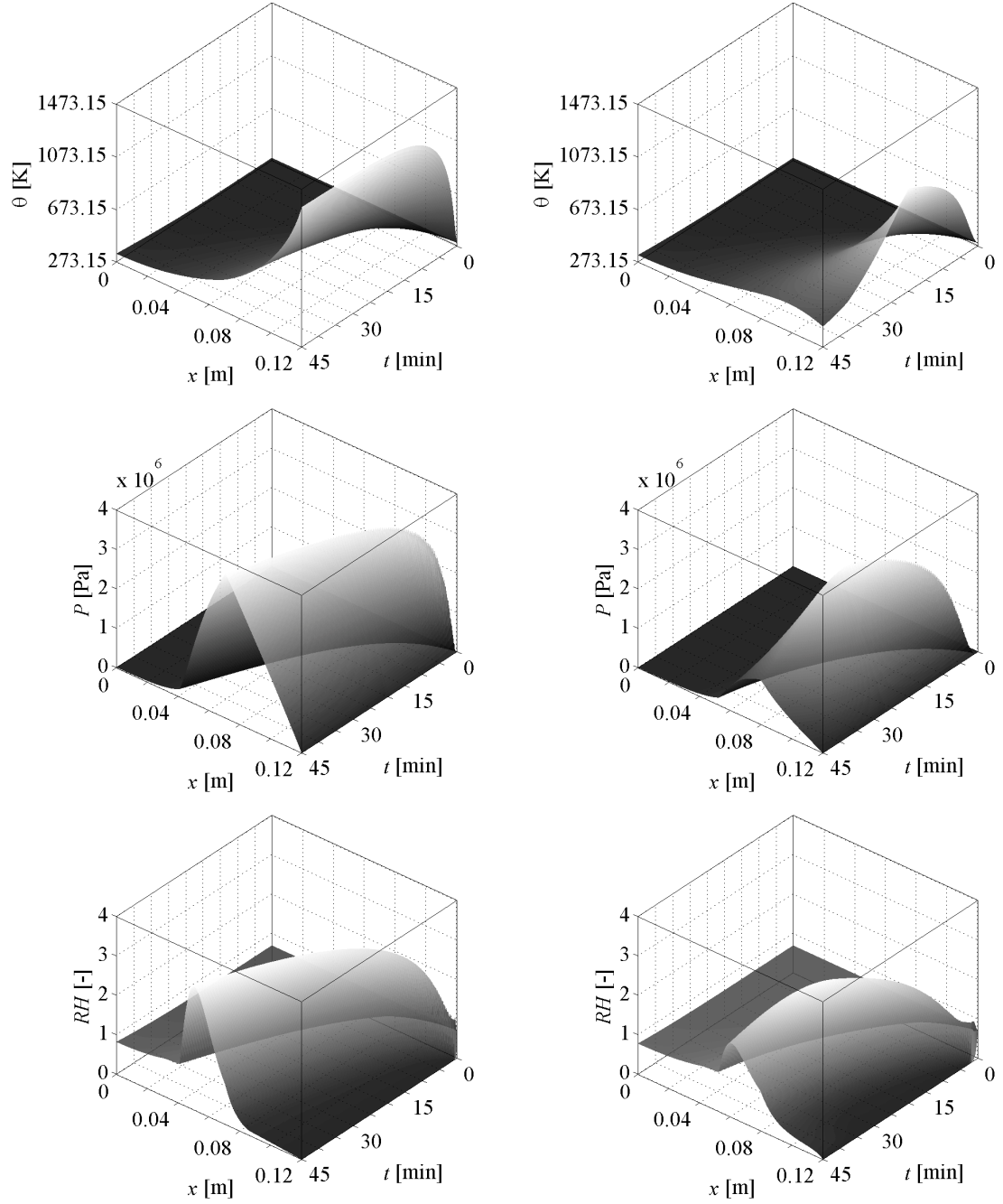


Figure 4: Spatial and time distribution of temperature, pore pressure and relative humidity for the analyzed wall exposed to: HC fire (left) and PM fire (right)

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